Journal of Approximation Theory **112**, 159–170 (2001)
doi:10.1006/jath.2001.3582, available online at http://www.idealibrary.com on IDEAL®

On Multivariate Quasipolynomials of the Minimal Deviation from Zero

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Communicated by Peter B. Borwein

Received April 14, 1998; accepted April 9, 2001

We generalize to several variables both the upper and the lower Gelfond bounds for the least uniform deviation from zero of the quasipolynomials (or Müntz–Legendre polynomials) on the segment $[0, 1]$. Orthonormal quasipolynomials are also considered.

Key Words: Müntz systems; Müntz–Legendre polynomials; orthogonal quasi-polynomials.

1. INTRODUCTION

The functions of the form

$$\sum_{i=0}^{m} a_i x^{\mu_i}, \tag{1}$$

where in general the values a_i and μ_i are complex numbers, are usually called Müntz–Legendre polynomials and were called quasipolynomials by A. O. Gelfond [7]. In this work, Gelfond obtains both lower and upper estimates for the least uniform deviation from zero of the real monic quasipolynomials ($0 \leqslant \mu_0 < \cdots < \mu_m$, $a_m = 1$, $a_i, \ \mu_i \in \mathbb{R}$) on $[0, 1]$. To do this, he finds the real monic quasipolynomial having minimal quadratic deviation from zero with respect to the weight function x^p, $p > -1$, as well as the value of such a minimal deviation. Some of the Gelfond's ideas were later used by E. Aparicio [1] to construct an orthonormal system of complex quasipolynomials with respect the weight function x^p, $p \geqslant 0$, in the interval $(0, 1)$ and under the assumption $\Re \mu_i > -\frac{1+p}{2}$. Further results

[1] Supported by the University of the Basque Country.

0021-9045/01 $35.00

concerning Müntz–Legendre polynomials were obtained in [2, 3, 8, 9]. A systematic account of many of such results is given in [4, 5].

In the present paper, the aforementioned results of Gelfond are extended to several variables. For simplicity, we shall only consider the two-dimensional case, since the extension to higher dimensions is straightforward. In Section 2 we give some properties of the orthonormal complex quasipolynomials on $(0, 1) \times (0, 1)$. In Section 3, following Gelfond's method, we obtain bounds for the value of the least uniform deviation from zero of the real monic quasipolynomials on $[0, 1] \times [0, 1]$.

2. ORTHONORMAL QUASIPOLYNOMIALS

Let $p_1, p_2 > -1$, and let $\mu_0^{(1)}, \mu_1^{(1)}, \ldots, \mu_{n_1}^{(1)}$ and $\mu_0^{(2)}, \mu_1^{(2)}, \ldots, \mu_{n_2}^{(2)}$ be two sequences of different complex numbers ordered in the following way: if $n < m$, then $|\theta_n| \leqslant |\theta_m|$ and, if $|\theta_n| = |\theta_m|$, then $\arg \theta_n < \arg \theta_m$. Moreover, we assume that, for each j, we have $\mu_j^{(i)} + \bar{\mu}_j^{(i)} + p_i + 1 > 0$, $i = 1, 2$, where, as usual, $\bar{\beta}$ denotes the conjugate of the complex number β.

We consider the bivariate quasipolynomials on the square $\mathscr{D} := (0, 1) \times (0, 1)$ having the form

$$P_{n_1, n_2}(x_1, x_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \alpha_{ij}^{(n_1, n_2)} x_1^{\mu_i^{(1)}} x_2^{\mu_j^{(2)}}, \qquad (x_1, x_2) \in \mathscr{D}, \tag{2}$$

where the coefficients $\alpha_{ij}^{(n_1, n_2)}$ are complex numbers, and $x_r^{\mu_s^{(r)}} = e^{\mu_s^{(r)} \ln x_r}$ $(\ln 1 = 0)$.

We wish to find an orthonormal system of quasipolynomials $\{P_{m,n}(x_1, x_2)\}$ on $\mathscr{D}$ with weight function $x_1^{p_1} x_2^{p_2}$ and with respect to the inner product

$$(P_{m,n}, P_{r,s}) = \iint_{\mathscr{D}} P_{m,n}(x_1, x_2)\, \bar{P}_{r,s}(x_1, x_2)\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2. \tag{3}$$

Let $\{P_m(x_1)\colon m = 0, 1, \ldots\}$ and $\{Q_n(x_2)\colon n = 0, 1, \ldots\}$ be two systems of orthogonal polynomials in one variable on the interval $(0, 1)$, with respect to the weight functions $x_1^{p_1}$ and $x_2^{p_2}$, respectively. Then, it is known that the direct product

$$\{P_m(x_1)\, Q_n(x_2) : m, n = 0, 1, \ldots\}$$

is a bivariate orthogonal system on the square $\mathscr{D}$, with respect to the weight function $x_1^{p_1} x_2^{p_2}$. Thus, from the formula for orthonormal Müntz–Legendre polynomials in one variable [1, 3], we can assert the following result in which we use the notations

$$a_{n_i+1}(x_i) := \prod_{j=0}^{n_i} (x_i - \mu_j^{(i)}), \qquad \bar{a}_{n_i+1}(x_i) := \prod_{j=0}^{n_i} (x_i - \bar{\mu}_j^{(i)}),$$
$$b_{n_i+1}(x_i) := \prod_{j=0}^{n_i} (x_i + \mu_j^{(i)} + p_i + 1), \qquad \bar{b}_{n_i+1}(x_i) := \prod_{j=0}^{n_i} (x_i + \bar{\mu}_j^{(i)} + p_i + 1), \tag{5}$$

for $i = 1, 2$.

Theorem 1. *The complex quasipolynomials*

$$\begin{aligned} R_{n_1, n_2}(x_1, x_2) &:= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \gamma_{ij}^{(n_1, n_2)} x_1^{\mu_i^{(1)}} x_2^{\mu_j^{(2)}} \\ &= \prod_{i=1}^{2} \Bigg[\sqrt{\mu_{n_i}^{(i)} + \bar{\mu}_{n_i}^{(i)} + p_i + 1} \\ &\quad \times \sum_{k=0}^{n_i} \frac{1}{(\mu_k^{(i)} + \bar{\mu}_{n_i}^{(i)} + p_i + 1)} \frac{\bar{b}_{n_i+1}(\mu_k^{(i)})}{a'_{n_i+1}(\mu_k^{(i)})} x_i^{\mu_k^{(i)}} \Bigg] \end{aligned} \tag{6}$$

$n_1 \geqslant 0$, $n_2 \geqslant 0$, form an orthonormal system, on the square $\mathscr{D}$, with weight function $x_1^{p_1} x_2^{p_2}$, $p_i \in (-1, \infty)$, $i = 1, 2$.

We remark that Theorem 1 can be directly derived from the orthonormality condition of the polynomials without making use of the results for one variable. To do this, we need the following lemma.

Lemma. *Let $e_{m,n}(x_1, x_2)$ be a monic polynomial of the form*

$$e_{m,n}(x_1, x_2) = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij} x_1^i x_2^j, \qquad \alpha_{mn} = 1, \quad \alpha_{ij} \in \mathbb{C}.$$

If

$$e_{m,n}(\mu_i, \nu_j) = 0, \qquad (i, j) \neq (m, n), \quad i = 0, \ldots, m, \quad j = 0, \ldots, n,$$

then we have

$$e_{m,n}(x_1, x_2) = \prod_{i=0}^{m-1} (x_1 - \mu_i) \prod_{j=0}^{n-1} (x_2 - \nu_j).$$

Proof. For each j, $0 \leqslant j \leqslant n-1$, we have

$$e_{m,n}(\mu_0, \nu_j) = e_{m,n}(\mu_1, \nu_j) = \cdots = e_{m,n}(\mu_m, \nu_j) = 0,$$

and, therefore, the polynomial $e_{m,n}(x_1, x_2)$ is a multiple of $\prod_{j=0}^{n-1} (x_2 - \nu_j)$. Likewise, the relations

$$e_{m,n}(\mu_i, \nu_0) = e_{m,n}(\mu_i, \nu_1) = \cdots = e_{m,n}(\mu_i, \nu_n) = 0, \qquad 0 \leqslant i \leqslant m-1,$$

imply that $e_{m,n}(x_1, x_2)$ is a multiple of $\prod_{i=0}^{m-1}(x_1-\mu_i)$. Therefore $e_{m,n}(x_1, x_2) = \prod_{i=0}^{m-1}(x_1-\mu_i)\prod_{j=0}^{n-1}(x_2-\nu_j)$. ▮

Proof of Theorem 1. Let $\{R^*_{n_1,n_2}(x_1, x_2) := \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\alpha_{ij}^{(n_1,n_2)} x_1^{\mu_i^{(1)}} x_2^{\mu_j^{(2)}}\}$ be a system of orthonormal complex quasipolynomials. Then

$$\iint_{\mathscr{D}} R^*_{n_1,n_2}(x_1, x_2)\, \bar{R}^*_{r_1,r_2}(x_1, x_2)\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2 = \begin{cases} 0 & \text{if} \quad (r_1, r_2) \neq (n_1, n_2) \\ 1 & \text{if} \quad (r_1, r_2) = (n_1, n_2), \end{cases} \tag{7}$$

and this implies that either

$$(R^*_{n_1,n_2}, x_1^{\bar{\mu}_{m_1}^{(1)}} x_2^{\bar{\mu}_{m_2}^{(2)}}) = \sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} \alpha_{l_1 l_2}^{(n_1,n_2)} \prod_{i=1}^{2} \frac{1}{\mu_{l_i}^{(i)} + \bar{\mu}_{m_i}^{(i)} + p_i + 1} = 0, \tag{8}$$

for $(m_1, m_2) \neq (n_1, n_2)$, $m_i = 0, ..., n_i$, $i = 1, 2$, or

$$\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} \bar{\alpha}_{l_1 l_2}^{(n_1,n_2)} \prod_{i=1}^{2} \frac{1}{\mu_{m_i}^{(i)} + \bar{\mu}_{l_i}^{(i)} + p_i + 1} = 0, \tag{9}$$

for $(m_1, m_2) \neq (n_1, n_2)$, $m_i = 0, ..., n_i$, $i = 1, 2$. To solve the system (8), we set

$$\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} \alpha_{l_1 l_2}^{(n_1,n_2)} \prod_{i=1}^{2} \frac{1}{\mu_{l_i}^{(i)} + x_i + p_i + 1} = C_{n_1 n_2} \frac{e_{n_1,n_2}(x_1, x_2)}{b_{n_1+1}(x_1)\, b_{n_2+1}(x_2)}, \tag{10}$$

where $C_{n_1 n_2} := \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\alpha_{ij}^{(n_1,n_2)}$, and $e_{n_1,n_2}(x_1, x_2)$ is a monic polynomial with complex coefficients of degree $\leqslant n_1$ in x_1, and of degree $\leqslant n_2$ in x_2, which vanishes at all the points $(\bar{\mu}_{m_1}^{(1)}, \bar{\mu}_{m_2}^{(2)})$ with $(m_1, m_2) \neq (n_1, n_2)$, $m_1 = 0, ..., n_1$, $m_2 = 0, ..., n_2$. By the preceding lemma, we have

$$e_{n_1,n_2}(x_1, x_2) = \prod_{i=1}^{2}\left[\prod_{j=0}^{n_i-1}(x_i - \bar{\mu}_j^{(i)})\right].$$

Then, (10) becomes

$$\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} \alpha_{l_1 l_2}^{(n_1,n_2)} \prod_{i=1}^{2} \frac{1}{\mu_{l_i}^{(i)} + x_i + p_i + 1} = C_{n_1 n_2} \prod_{i=1}^{2} \frac{\prod_{j=0}^{n_i-1}(x_i - \bar{\mu}_j^{(i)})}{b_{n_i+1}(x_i)}. \tag{11}$$

Multiplying (11) by $\prod_{i=1}^{2}(\mu_{l_i}^{(i)} + x_i + p_i + 1)$, and taking $x_i = -\mu_{l_i}^{(i)} - p_i - 1$, $i = 1, 2$, we obtain, for $0 \leqslant l_1 \leqslant n_1$ and $0 \leqslant l_2 \leqslant n_2$,

$$\alpha_{l_1 l_2}^{(n_1,n_2)} = C_{n_1 n_2} \prod_{i=1}^{2}\left[\frac{1}{(\mu_{l_i}^{(i)} + \bar{\mu}_{n_i}^{(i)} + p_i + 1)} \frac{\bar{b}_{n_i+1}(\mu_{l_i}^{(i)})}{a'_{n_i+1}(\mu_{l_i}^{(i)})}\right], \tag{12}$$

where the product $\prod_{t=0,\, t\neq k}^{m}(x_k - x_t)$ has been written in the form $[\prod_{t=0}^{m}(x - x_t)]'_{x=x_k}$. From the normality condition, we get

$$\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\bar{\alpha}_{l_1 l_2}^{(n_1,n_2)}\prod_{i=1}^{2}\frac{1}{\mu_{n_i}^{(i)}+\bar{\mu}_{l_i}^{(i)}+p_i+1}=\frac{1}{\alpha_{n_1 n_2}^{(n_1,n_2)}}. \tag{13}$$

Taking complex conjugates in (11), substituting (x_1, x_2) by $(\mu_{n_1}^{(1)}, \mu_{n_2}^{(2)})$ and bearing in mind (13), we obtain

$$\frac{1}{\alpha_{n_1 n_2}^{(n_1,n_2)}}=\bar{C}_{n_1 n_2}\prod_{i=1}^{2}\frac{a'_{n_i+1}(\mu_{n_i}^{(i)})}{\bar{b}_{n_i+1}(\mu_{n_i}^{(i)})}. \tag{14}$$

Letting $(l_1, l_2) = (n_1, n_2)$ in (12) and multiplying by (14), we have

$$1=|C_{n_1 n_2}|^2\frac{1}{\prod_{i=1}^{2}(\mu_{n_i}^{(i)}+\bar{\mu}_{n_i}^{(i)}+p_i+1)},$$

that is

$$|C_{n_1 n_2}|=\sqrt{\prod_{i=1}^{2}(\mu_{n_i}^{(i)}+\bar{\mu}_{n_i}^{(i)}+p_i+1)}\,. \tag{15}$$

Then, (6) follows from the fact that $R_{n_1,n_2}(x_1, x_2) = (|C_{n_1 n_2}|/C_{n_1 n_2})\, R^*_{n_1,n_2}(x_1, x_2)$. ∎

From the general theory of orthonormal polynomials, we also have the following result, where $\tilde{R}_{n_1,n_2}(x_1, x_2)$ stands for the monic quasipolynomial associated with $R_{n_1,n_2}(x_1, x_2)$, that is $\tilde{R}_{n_1,n_2}(x_1, x_2) := (1/\gamma_{n_1 n_2}^{(n_1,n_2)})\, R_{n_1,n_2}(x_1, x_2)$.

THEOREM 2. *If $P_{n_1,n_2}(x_1, x_2)$ is a monic quasipolynomial of the type* (2), *then the integral*

$$\iint_{\mathscr{D}} |P_{n_1,n_2}(x_1, x_2)|^2\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2$$

is a minimum if and only if $P_{n_1,n_2}(x_1, x_2) = \tilde{R}_{n_1,n_2}(x_1, x_2)$. Moreover,

$$\begin{aligned}\iint_{\mathscr{D}} &|\tilde{R}_{n_1,n_2}(x_1, x_2)|^2\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2\\ &=\frac{1}{|\gamma_{n_1 n_2}^{(n_1,n_2)}|^2}\iint_{\mathscr{D}} |R_{n_1,n_2}(x_1, x_2)|^2\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2\\ &=\prod_{i=1}^{2}\left[(\mu_{n_i}^{(i)}+\bar{\mu}_{n_i}^{(i)}+p_i+1)\frac{|a'_{n_i+1}(\mu_{n_i}^{(i)})|^2}{|\bar{b}_{n_i+1}(\mu_{n_i}^{(i)})|^2}\right],\end{aligned}$$

where $a_{n_i+1}(x_i)$ and $\bar{b}_{n_i+1}(x_i)$ are the same as in (5).

Remark 1. Theorems 2 and 1 generalize results of Gelfond [7] and Aparicio [1].

3. MINIMAL UNIFORM DEVIATION

In this section, we obtain both upper and lower bounds for the least uniform deviation from zero of the real monic quasipolynomials on $\bar{\mathscr{D}} = [0, 1] \times [0, 1]$. We state the following.

THEOREM 3. *For $i = 1, 2$, let $p_i > -1$, let $0 \leqslant \mu_0^{(i)} < \mu_1^{(i)} < \cdots < \mu_{n_i}^{(i)}$, and set*

$$M_{n_1 n_2} := \inf_{R_{n_1, n_2} \in H} \max_{(x_1, x_2) \in \bar{\mathscr{D}}} |R_{n_1, n_2}(x_1, x_2)|,$$

where H denotes the set, of all real monic quasipolynomials of the form (2). *Then:*

$$M_{n_1 n_2} \geqslant \prod_{i=1}^{2} \left[(1+p_i)^{1/2} (2\mu_{n_i}^{(i)} + p_i + 1)^{1/2} \frac{\prod_{s=0}^{n_i - 1} (\mu_{n_i}^{(i)} - \mu_s^{(i)})}{\prod_{s=0}^{n_i} (\mu_{n_i}^{(i)} + \mu_s^{(i)} + p_i + 1)} \right] \tag{16}$$

and

$$M_{n_1 n_2} \leqslant \max_{(x_1, x_2) \in \bar{\mathscr{D}}} \min\{A_1(x_1, x_2), A_2(x_1, x_2)\} \times \prod_{j=1}^{2} \frac{\prod_{s=0}^{n_j - 1} (\mu_{n_j}^{(j)} - \mu_s^{(j)})}{\prod_{s=0}^{n_j} (\mu_{n_j}^{(j)} + \mu_s^{(j)} + p_j + 1)}, \tag{17}$$

where, for $(x_1, x_2) \in \mathscr{D}$,

$$A_1(x_1, x_2) := \prod_{j=1}^{2} \left\{ (2\mu_{n_j}^{(j)} + p_j + 1)\, x_j^{-(1+p_j)/2} \times \sqrt{2 \sum_{k=0}^{n_j} \mu_k^{(j)} + (n_j + 1)(p_j + 1)} \right\} \tag{18}$$

and

$$A_2(x_1, x_2) := \prod_{j=1}^{2} \left\{ C_j \left(1 + \frac{p_j+1}{\mu_{n_j}^{(j)}}\right) (p_j+2)^{q_j} \right.$$
$$\times \left[1 + \frac{1}{\delta_j \ln(1/x_j)} (\mu_{n_j}^{(j)} + \ln(1/\delta_j) + n_j \ln \mu_{n_j}^{(j)}) \right]$$
$$\left. \times \exp[(1+\varepsilon_j)(2\varepsilon_j + p_j + 1)\, \rho_{n_j}] \right\} \tag{19}$$

and where, for $j = 1, 2$,

$$\rho_{n_j} := \sum_{k=1}^{n_j} \frac{1}{\mu_k^{(j)}},$$
$$\varepsilon_j = \mu_0^{(j)} + \delta_j \qquad (0 < \delta_j < \min[1, (\mu_1^{(j)} - \mu_0^{(j)})/2]),$$

q_j *being the natural number determined by the inequalities*

$$\mu_{q_j-1}^{(j)} - \mu_1^{(j)} < 1 \leqslant \mu_{q_j}^{(j)} - \mu_1^{(j)} < \mu_{q_j}^{(j)} - \varepsilon_j$$

and $C_j > 0$ *being a constant independent of* n_j, x_j, δ_j *and* p_j.

Proof. Let $S_{n_1, n_2}(x_1, x_2) := \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{ij} x_1^{\mu_i^{(1)}} x_2^{\mu_j^{(2)}}$ be a real monic quasi-polynomial satisfying the condition of minimum

$$\min_{b_{ij} \in \mathbb{R},\, b_{n_1 n_2} = 1} \iint_{\mathscr{D}} \left(\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} b_{ij} x_1^{\mu_i^{(1)}} x_2^{\mu_j^{(2)}} \right)^2 x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2$$
$$= \iint_{\mathscr{D}} S_{n_1, n_2}^2(x_1, x_2)\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2,$$

with $p_i > -1$, $i = 1, 2$. Then, by Theorem 2,

$$S_{n_1, n_2}(x_1, x_2) = \prod_{i=1}^{2} \left[(2\mu_{n_i}^{(i)} + p_i + 1) \frac{a'_{n_i+1}(\mu_{n_i}^{(i)})}{b_{n_i+1}(\mu_{n_i}^{(i)})} \right.$$
$$\left. \times \sum_{r=0}^{n_i} \frac{1}{(\mu_r^{(i)} + \mu_{n_i}^{(i)} + p_i + 1)} \frac{b_{n_i+1}(\mu_r^{(i)})}{a'_{n_i+1}(\mu_r^{(i)})} x_i^{\mu_r^{(i)}} \right]$$
$$= \prod_{i=1}^{2} \left[(2\mu_{n_i}^{(i)} + p_i + 1) \frac{a'_{n_i+1}(\mu_{n_i}^{(i)})}{b_{n_i+1}(\mu_{n_i}^{(i)})} \right]$$
$$\times \iint_{\mathscr{D}} \phi(x_1 u_1, x_2 u_2)\, u_1^{\mu_{n_1}^{(1)} + p_1} u_2^{\mu_{n_2}^{(2)} + p_2}\, du_1\, du_2$$
$$= \prod_{i=1}^{2} \left[(2\mu_{n_i}^{(i)} + p_i + 1) \frac{a'_{n_i+1}(\mu_{n_i}^{(i)})}{b_{n_i+1}(\mu_{n_i}^{(i)})} \int_0^1 \phi_i(x_i u_i)\, u_i^{\mu_{n_i}^{(i)} + p_i}\, du_i \right], \tag{20}$$

where

$$\phi(t_1, t_2) := \prod_{i=1}^{2} \left[\sum_{r=0}^{n_i} \frac{b_{n_i+1}(\mu_r^{(i)})}{a'_{n_i+1}(\mu_r^{(i)})} t_i^{\mu_r^{(i)}} \right] = \phi_1(t_1)\, \phi_2(t_2),$$

$$\phi_i(t_i) := \sum_{r=0}^{n_i} \frac{b_{n_i+1}(\mu_r^{(i)})}{a'_{n_i+1}(\mu_r^{(i)})} t_i^{\mu_r^{(i)}} \qquad (i = 1, 2).$$

Moreover,

$$\iint_{\mathscr{D}} S^2_{n_1, n_2}(x_1, x_2)\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2 = \prod_{i=1}^{2} (2\mu_{n_i}^{(i)} + p_i + 1) \left[\frac{a'_{n_i+1}(\mu_{n_i}^{(i)})}{b_{n_i+1}(\mu_{n_i}^{(i)})} \right]^2. \tag{21}$$

Therefore, denoting by $T_{n_1, n_2}(x_1, x_2)$ the quasipolynomial in H such that

$$M_{n_1 n_2} = \max_{(x_1, x_2) \in \bar{\mathscr{D}}} |T_{n_1, n_2}(x_1, x_2)|,$$

we have

$$\begin{aligned} \iint_{\mathscr{D}} S^2_{n_1, n_2}(x_1, x_2)\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2 &\leqslant \iint_{\mathscr{D}} T^2_{n_1, n_2}(x_1, x_2)\, x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2 \\ &\leqslant \frac{1}{p_1+1} \frac{1}{p_2+1} M^2_{n_1 n_2}, \end{aligned}$$

and (16) follows from (21).

To show (17), let $(x_1, x_2) \in \mathscr{D}$. Consider the integral

$$\begin{aligned} &\frac{-1}{4\pi^2} \iint_{\mathscr{C}_1 \times \mathscr{C}_2} \prod_{j=1}^{2} \left\{ x_j^{z_j} \frac{d}{dz_j} \left[\frac{1}{z_j + \mu_{n_j}^{(j)} + p_j + 1} \frac{b_{n_j+1}(z_j)}{a_{n_j+1}(z_j)} \right] dz_j \right\} \\ &= \prod_{j=1}^{2} \left\{ \frac{-1}{2\pi i} \int_{\mathscr{C}_j} x_j^{z_j} \frac{d}{dz_j} \left[\frac{1}{z_j + \mu_{n_j}^{(j)} + p_j + 1} \frac{b_{n_j+1}(z_j)}{a_{n_j+1}(z_j)} \right] dz_j \right\} \\ &= \prod_{j=1}^{2} \left[\ln x_j \sum_{r=0}^{n_j} \frac{b_{n_j+1}(\mu_r^{(j)})}{a'_{n_j+1}(\mu_r^{(j)})} \frac{x_j^{\mu_r^{(j)}}}{\mu_{n_j}^{(j)} + \mu_r^{(j)} + p_j + 1} \right], \end{aligned} \tag{22}$$

where $\mathscr{C}_j := \{z_j \colon |z_j - \mu_{n_j}^{(j)}/2| = \mu_{n_j}^{(j)}/2 + \alpha_j,\ 0 < \alpha_j < 1/2\}$, $j = 1, 2$. Let ε_j be such that $\mu_0^{(j)} < \varepsilon_j < (\mu_0^{(j)} + \mu_1^{(j)})/2$, $j = 1, 2$. Separate the first term in the sum, and rewrite the sum $\sum_{r=1}^{n_j}$ as a complex integral over a semicircle Γ_j with diameter on $\Re z_j = \varepsilon_j$ and centre at $z_j = \varepsilon_j$, surrounding the poles $\mu_1^{(j)}, \mu_2^{(j)}, \ldots, \mu_{n_j}^{(j)}$. Then, by (20) and (22), we obtain

$$\prod_{j=1}^{2} \int_0^1 \phi_j(x_j u_j)\, u_j^{\mu_{n_j}^{(j)} + p_j}\, du_j$$

$$= \prod_{j=1}^{2} \left\{ -\frac{b_{n_j+1}(\mu_0^{(j)})}{a'_{n_j+1}(\mu_0^{(j)})} \frac{x_j^{\mu_0^{(j)}}}{\mu_{n_j}^{(j)} + \mu_0^{(j)} + p_j + 1} + \frac{1}{2\pi i \ \ln x_j} \int_{\varepsilon_j - i\infty}^{\varepsilon_j + i\infty} x_j^{z_j} \frac{d}{dz_j} \left[\frac{1}{z_j + \mu_{n_j}^{(j)} + p_j + 1} \frac{b_{n_j+1}(z_j)}{a_{n_j+1}(z_j)} \right] dz_j \right\},$$

where $z_j = \varepsilon_j + i y_j$. From this and from the estimate given in [7] for one variable, we have

$$\left| \int\int_{\mathscr{D}} \phi(x_1 u_1, x_2 u_2)\, u_1^{\mu_{n_1}^{(1)} + p_1} u_2^{\mu_{n_2}^{(2)} + p_2}\, du_1\, du_2 \right|$$

$$= \left| \prod_{j=1}^{2} \int_0^1 \phi_j(x_j u_j)\, u_j^{\mu_{n_j}^{(j)} + p_j}\, du_j \right|$$

$$< \prod_{j=1}^{2} \left\{ C'_j \frac{(p_j + 2)^{q_j}}{\mu_{n_j}^{(j)}} \left[1 + \frac{1}{\delta_j \ln(1/x_j)} (\mu_{n_j}^{(j)} + \ln(1/\delta_j) + n_j \ln \mu_{n_j}^{(j)}) \right] \times \exp[(1 + \varepsilon_j)(2\varepsilon_j + p_j + 1)\, \rho_{n_j}] \right\}, \tag{23}$$

where ρ_{n_j}, ε_j, δ_j and q_j are the same as in the statement of the theorem, and the constant $C'_j > 0$ is independent of n_j, x_j, δ_j and p_j. From (20) and (23), it follows that

$$|S_{n_1, n_2}(x_1, x_2)| < A_2(x_1, x_2) \prod_{j=1}^{2} \frac{a'_{n_j+1}(\mu_{n_j}^{(j)})}{b_{n_j+1}(\mu_{n_j}^{(j)})}, \tag{24}$$

where $A_2(x_1, x_2)$ is defined in (19). This provides an upper bound for $M_{n_1 n_2}$ useful when both x_i are not close to 1.

Next, we give another bound for $|S_{n_1, n_2}(x_1, x_2)|$ appropriate for values of x_i close to 1. Following Gelfond [7], we have

$$\left| \int\int_{\mathscr{D}} \phi(x_1 u_1, x_2 u_2)\, u_1^{\mu_{n_1}^{(1)} + p_1} u_2^{\mu_{n_2}^{(2)} + p_2}\, du_1\, du_2 \right|$$

$$< \prod_{j=1}^{2} \left\{ x_j^{-(1+p_j)/2} \sqrt{2 \sum_{k=0}^{n_j} \mu_k^{(j)} + (n_j + 1)(p_j + 1)} \right\}. \tag{25}$$

This, together with (20), yields

$$|S_{n_1, n_2}(x_1, x_2)| < A_1(x_1, x_2) \prod_{j=1}^{2} \frac{a'_{n_j+1}(\mu_{n_j}^{(j)})}{b_{n_j+1}(\mu_{n_j}^{(j)})}, \tag{26}$$

where $A_1(x_1, x_2)$ is defined in (18).

The inequality (17) follows from (24), (26) and the fact that

$$M_{n_1 n_2} \leqslant \max_{(x_1, x_2) \in \bar{\mathscr{D}}} |S_{n_1, n_2}(x_1, x_2)|.$$

This completes the proof of the theorem. ▮

Remark 2. It is clear that the function $A_1(x_1, x_2)$ (resp. $A_2(x_1, x_2)$) is decreasing (resp. increasing) in each variable separately. Therefore, the function $\min\{A_1(\cdot, \cdot), A_2(\cdot, \cdot)\}$ is bounded on $\bar{\mathscr{D}}$.

Remark 3. For $n_2 = 0$, Gelfond's Theorem [7] is obtained.

The following corollary is a particular case of Theorem 3.

COROLLARY. *If* $\mu_i^{(1)} = i^{h_1}$, $\mu_j^{(2)} = j^{h_2}$, $0 < h_1, h_2 < 1$, *then*

$$M_{n_1 n_2} \geqslant \prod_{j=1}^{2} [\sqrt{2c_j} + o(1)]\, n_j^{h_j - 1/2} N_{n_1 n_2} \tag{27}$$

and

$$M_{n_1 n_2} \leqslant \prod_{i=1}^{2} K_i n_i^{\gamma_i} N_{n_1 n_2}, \tag{28}$$

where

$$c_j := \left(2 \int_0^1 \frac{dx_j}{1 + x_j^{h_j}}\right)^{-1}, \tag{29}$$

$$N_{n_1 n_2} := \prod_{j=1}^{2} \frac{\prod_{s=0}^{n_j - 1} (n_j^{h_j} - s_j^{h_j})}{\prod_{s=0}^{n_j} (n_j^{h_j} + s_j^{h_j} + p_j + 1)}$$
$$= \exp\left\{\prod_{j=1}^{2} \left[-n_j \int_0^1 \ln \frac{1 + z_j^{h_j}}{1 - z_j^{h_j}}\, dz_j + o(n_j)\right]\right\},$$

$$\gamma_i > \max\left(1, \frac{3}{2} h_i + \frac{1}{2}\right)$$

and the constant $K_i > 0$ *does not depend upon* n_i.

Proof. In view of the assumptions on the exponents $\mu_i^{(h)}$, the inequality (27) follows from (16) on taking $p_i+1=c_i n_i^{h_i-1}$, where c_i is given in (29). Further, the inequality (28) follows from (17) on taking $p_i+1=\varepsilon_i=\delta_i=n_i^{h_i-1}$ and $\ln(1/x_i)=n_i^{1-h_i}$, $i=1, 2$. ▮

Remark 4. It should be observed that, for large enough n_i, the values of p_i+1 used in the above proof to show (27) are close to the values of p_i+1 for which the maximum of the right-hand side in (16) is achieved [7].

4. CONCLUDING REMARKS

Remark 5. It is known [6] that the monic bivariate polynomial $T_{n,m}(x, y)=\sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}x^i y^j$ having minimal uniform deviation from zero on $\bar{\mathscr{D}}$ is

$$T_{n,m}(x, y)=T_n(x)\,T_m(y),$$

where $T_k(\cdot)$ is the monic Chebyshev polynomial on the segment $[0, 1]$. Therefore, if $\mu_i^{(1)}=i$ and $\mu_j^{(2)}=j$, one finds that the two components of the inequality (16) have the same order (see [7]).

Remark 6. In the real case, Eqs. (20) and (21) can also be obtained in the following way. Consider the function of the $(n_1+1)(n_2+1)-1$ variables $v_{00}, v_{01}, \ldots, v_{n_1 n_2-1}$ given by

$$\Phi(v_{00}, v_{01}, \ldots, v_{n_1 n_2-1}) := \iint_{\mathscr{D}} \left(\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} v_{ij} x_1^{\mu_i^{(1)}} x_2^{\mu_j^{(2)}} \right)^2 x_1^{p_1} x_2^{p_2}\, dx_1\, dx_2,$$

with $v_{n_1 n_2}=1$. If this function attains its minimum value at $(a_{00}, a_{01}, \ldots, a_{n_1 n_2-1})$, we have

$$\frac{1}{2}\frac{\partial \Phi}{\partial v_{r_1 r_2}}(a_{00}, a_{01}, \ldots, a_{n_1 n_2-1}) = \sum_{l_1=0}^{n_1} \sum_{l_2=0}^{n_2} a_{l_1 l_2} \prod_{i=1}^{2} \left[\frac{1}{\mu_{l_i}^{(i)}+\mu_{r_i}^{(i)}+p_i+1} \right] = 0,$$

$a_{n_1 n_2}=1$, $(r_1, r_2) \neq (n_1, n_2)$, $r_i=0, \ldots, n_i$, $i=1, 2$. This equation is the same as (8). Thus, formulas (20) and (21) follow by the same argument as in the proof of Theorem 1.

Remark 7. Other representations for the quasipolynomials (6) are

$$R_{n_1, n_2}(x_1, x_2) = \prod_{j=1}^{2} \left[\frac{(\mu_{n_j}^{(j)} + \bar{\mu}_{n_j}^{(j)} + p_j + 1)^{1/2}}{2\pi i \ \ln(1/x_j)} \right]$$

$$\times \iint_{\gamma_1 \times \gamma_2} \prod_{j=1}^{2} \left\{ x_j^{u_j} \frac{d}{du_j} \left[\frac{1}{u_j + \bar{\mu}_{n_j}^{(j)} + p_j + 1} \frac{\bar{b}_{n_j+1}(u_j)}{a_{n_j+1}(u_j)} \right] du_j \right\}$$

and

$$R_{n_1, n_2}(x_1, x_2) = \prod_{j=1}^{2} \left[\frac{(\mu_{n_j}^{(j)} + \bar{\mu}_{n_j}^{(j)} + p_j + 1)^{1/2}}{2\pi i} \right]$$

$$\times \iint_{\gamma_1 \times \gamma_2} \prod_{j=1}^{2} \left[x_j^{u_j} \frac{\bar{b}_{n_j+1}(u_j)}{(u_j + \bar{\mu}_{n_j}^{(j)} + p_j + 1)\, a_{n_j+1}(u_j)} du_j \right],$$

where the simple contour γ_j, $j = 1, 2$, lies completely to the right of the vertical line $\Re u_j = -1/2$, and surrounds all the zeros of the denominator in the integrand. The last representation is the bidimensional analogue of formula (2.9) in [3].

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